Numerical Analysis

1 Introduction 09/30

Today, we learnt the basic errors in the numerical analysis

- Absolute error and Relative error
  - $\delta x/x$
  - relative error is related with a significant figures
  - relative error increases when x is close to 0
- Rounding error
  - It is the difference between the result produced by a given algorithm using exact arithmetic and the result produced by the same algorithm using finite-precision, rounded arithmetic.
- Loss of significance
  - It occurs when an operation on two numbers increases relative error substantially more than it increases absolute error, for example in subtracting two nearly equal numbers such as quadratic formula
- Numerical differentiation
  - $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$
  - It is used for debug of the differentiation.
  - $fl(f(x)) = f(x)(1+\delta)$ ($|\delta| \leq u$)
  - We determine the value of h within the balance between truncation error and rounding error

2 Nonlinear equation 10/07

Today, we learned how to analyze nonlinear equations in computer. Three ways for one variable equation are introduced in the lecture. Furthermore, Newton's method is introduced. It can be applied to n variables using Jacobi matrix.

- if it includes only one variable
  - Bisection method
    * It is generaly used.
  - regula falsi
    * It is sometimes faster than bisection method

- Golden-section search
    - method for search local minimum value
- Newton's method
  -
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
  - quadratic convergence
- Homotopy method

3 LU decomposition 10/21

- we could regard to solve simultaneous linear equations as $A\mathbf{x} = \mathbf{b}$
- Gaussian elimination
  - how to it works
  - program
- LU decomposition
  - by seeing k steps of gaussian elimination as a matrix calculation, we coulg gain $A = L_k A$
  - $A = L_1^{-1} L_2^{-1} ...... L_{n-1}^{-1} U$, which is a shape of the LU decomposition.
  - If a matrix can be LU decomposed, its decompositon is unique
  - Matrix A is LU decomposed if and only if the determinant of A's principal submatrix is not equal to 0
  - crout's method
  - inner-product form

4 Ordinary differential equation 10/28

- To solve $\frac{dy}{dx} = f(x, y)$
- discrete variable method
- Euler method
  - $y_{n+1} = y_n + hf(x_n, y_n)$
  - global error is about $|y_{\frac{x}{h}} - y(x)| = O(h)$ under the followintg two conditions
    - f's Lipschitz condition $|f(x, y) - f(x, z)| \leq L|y - z|$
    - twice defferentiable $|y''(x)| \leq 2M$
- Runge-Kutta method
  - $y_{n+1} = y_n + h \sum_{i=1}^{s} b_i k_i$
  - $k_i = f(x_n + c_i h, y_n + h \sum_{i=1}^{s} a_{ij} k_j)$
- Stability region

5 LU decomposition and simultaneous linear equations 11/7

5.1 LU decomposition

- if A is band matrix, L and U are the same size band matrix
- Cholesky decomposition
- LDU decomposition

5.2 Simultaneous linear equations

- definition of Norm of the matrix
- When solving simultaneous linear equations in computers, we will surely have an error, which is in proportion to condition numbers of A (E can be calculated by using $\hat{x}$)

$$\frac{|\mathbf{x} - \hat{\mathbf{x}}|}{|\hat{\mathbf{x}}|} \leq |A^{-1}||A|\frac{|E|}{|A|} = cond(A)\frac{|E|}{|A|}$$

- gastinel theorem

6 CG method 11/11

- Data structures for including the information of the sparse matrix
- Iteretive method
  - Jacobi methods
  - Gauss-Seidel methods
  - SOR methods
- Conjugate gradient method is used to solve simultaneous linear equations, especially when A is sparse, symmetric, and positive-definite.

7 Partial differential equation 11/13

7.1 Three kinds of partial differential equation

- convection diffusion equation $\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$
- diffusion equation $\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial x^2}$
- convection diffusion equation

7.2 How to solve diffusion equation and convection differential equation

- consider space as a matrix and approximate the partial differenciation

7.3 von neumann stability analysis

- hypothesize $u_{j,k} = \alpha^k e^{i\beta jt}$

8 CG method and eigenvalue of matrix 11/18

- CG algorithm is faster when A is similar to the identity matrix, so we can apply some matrix to A in order to make it be similar to the identity matrix.
- Power iteration, which is used to find the maximum eigenvalue.
- Jacobi method
  - convert $A\mathbf{x} = \lambda\mathbf{x}$ into $QAQ^\top Q\mathbf{x} = \lambda Q\mathbf{x}$
  - decide Q making $QAQ^\top$ be the diagonal matrix

9 Householder conversions 11/25

- if we choose carefully i and j in the Jacobi method, it will necessarily converge, so we should choose i and j properly.
- Three ways to choose i and j
- plus and cons of Jacobi methods
- Householder conversion
  - House holder matrix : $H(\mathbf{w}) = \mathbf{I} - \mathbf{w}^\top\mathbf{w}$
  - Tridiagonalization using House holder matrix
- Gershgorin theorem
- Strum theorem

10 Singular value decomposition 12/02

- Any matrix A can be written in a form $A = W\Sigma V^\top$ under W and V are orthogonal matrix.
  - Proof of the singular value decomposition
  - use householder matrix to calculate the singular value
- relation with simultaneous linear equations
  - consider the minimization of $|A\mathbf{x} - \mathbf{b}|$
  - we gain $\mathbf{x} = V'diag(\frac{1}{\sigma_1}, ..., \frac{1}{\sigma_k})W'\mathbf{b}$
  - using generalized inverse matrix A', we obtain the answer as $\mathbf{x} = A'\mathbf{b}$
- other application
  - linear model
  - polar decomposition of the matrix

11 Poisson equation 12/09

- Poisson equation is
$$-(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x^2}) = f(x, y)$$
- two types of boundary conditions
  - Dirichlet condition $u(x, 0) = a(x)$
  - Neumann condition $\frac{\partial u(x,t)}{\partial y} = b(x)$
- first discritize the space
- approximate partial differentiation with discritized space
- Translate it into matrix and vectors ($A\mathbf{u} = \mathbf{b}$)
- solve liniar equation
- check if $\mathbf{u}$is correct
  - compatibility, stability, and convergence

12 FFT 12/16

- DFT
  - calculate $p(w^h)$, under $w = \exp(\frac{-2\pi i}{k})$
- Algorithm of FFT
- Richardson process
- Aitken process

13 Random number 12/23

13.1 random number

- linear congruential generator
  - $x_n = ax_{n-1} + c(mod m)$
- Additive lagged Fibonacci
  - $x_n = x_{n-j} + x_{n-k}(mod 2^m)$
- Mersenne Twister

13.2 application

- inversion method
- rejection sampling
- Box-Muller method
  - number sampling method for generating normally distributed random numbers

- Monte Carlo method
  - for calculating integral